DYNAMIC PROBLEM FOR A TWO-LAYER MEDIUM WITH A VIBRATIONAL SOURCE AT THE BOUNDARY INTERFACE

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The present study is concerned with an analysis of gravitational and acoustic waves which are excited by a vibrational source deeply placed in a liquid covered by ice. An analysis of the rigidity characteristics of ice modeled by an elastic layer or by a Kirchhoff plate is done by factorization of the solution to the integral equation equivalent to an initially combined boundary value problem. The uncombined boundary condition is used to solve problems for unrestricted ice fields in [1-3], whereas combined conditions with vibrational sources positioned at the boundary of the medium are used in [4].

1. We will consider the excitation of a wave field in a layer of liquid covered by an elastic layer, where sources of harmonic vibrations are positioned at the boundary interface of the media. The source is modeled as a discontinuity in the distributed load and is assigned over the range x, y $\in \Omega$, z = $-C$. A layer of ideal, heavy liquid ($|x, y| \leqslant \infty$, $-H \leqslant z \leqslant -C$) serves as an absolutely rigid foundation. The field of displacements in the elastic layer ($|x, y| \leqslant \infty$, $-C \leqslant z \leqslant 0$) is described by Lame's equations of motion, and the potential of the velocities of the liquid particles satisfies the wave equation. The time dependences of the above functions are given by the relation $f_0(x, y, z, t) = f(x, y, z)e^{-i\omega t}$. A vibrational source is positioned at the boundary interface of the two media, and the stresses and the normal velocities are equal outside the region occupied by the source. Inside this region (x, y $\in \Omega$, z = $-C$), the normal stresses at the edges are $\sigma^{\mathrm{I}}(x, y) = \sigma^*(x, y, -C + 0)$, $\sigma^{\mathrm{II}}(x, y) = \sigma^*(x, y, -C - 0)$, and, therefore, $\Delta\sigma(x, y) = \sigma^{\mathrm{I}}(x, y) - \sigma^{\mathrm{II}}(x, y)$. Because of the radiation conditions of the wave, the formulation of the problem does not apply at infinity.

One can use integral transformations to reduce the boundary value problem to the solution of an integral equation in terms of the unknown change in the velocities of the displacements $\Delta V_z^*(x, y) = V^*(x, y, -C + 0) - V^*(x, y, -C - 0)$ at the edges of the vibrational source

$$\iint_{\Omega} \Delta V_z(\xi, \eta)\, k(x-\xi, y-\eta)\, d\xi d\eta = f(x, y), \quad x, y \in \Omega, \tag{1.1}$$
$$f(x, y) = \sigma(x, y) + \iint_{\Omega} \Delta\sigma(\xi, \eta)\, m(x-\xi, y-\eta)\, d\xi d\eta,$$
$$k(t, s) = \frac{1}{4\pi^2}\int_{\Gamma_1}\int_{\Gamma_2} K(u)\, e^{-i(\alpha t+\beta s)} d\alpha d\beta,$$
$$m(t, s) = \frac{1}{4\pi^2}\int_{\Gamma_1}\int_{\Gamma_2} M(u)\, e^{-i(\alpha t+\beta s)} d\alpha d\beta.$$

Using dimensionless functions and variables, Eq.(1.1) has the form

$$K(u) = i[m\gamma_0 \operatorname{sh}(\gamma_0(h-c)) - \varkappa^2 \operatorname{ch}(\gamma_0(h-c))]\Delta_0(u)/\Delta(u), \tag{1.2}$$
$$M(u) = [m\gamma_0 \operatorname{sh}(\gamma_0(h-c)) - \varkappa^2 \operatorname{ch}(\gamma_0(h-c))]\Delta_1(u)/\Delta(u),$$
$$\Delta_0(u) = 4\left[(\gamma^4 + \gamma_1^2\gamma_2^2u^4)\operatorname{sh}(\gamma_1 c)\operatorname{sh}(\gamma_2 c) - 2\gamma_1\gamma_2u^2\gamma^2(\operatorname{ch}(\gamma_1 c)\operatorname{ch}(\gamma_2 c) - 1)\right],$$
$$\Delta_1(u) = \varkappa^2\gamma_1[\gamma^2\operatorname{sh}(\gamma_2 c)\operatorname{ch}(\gamma_1 c) - u^2\gamma_1\gamma_2\operatorname{sh}(\gamma_1 c)\operatorname{ch}(\gamma_2 c)],$$

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$$\Delta(u) = \Delta_1(u)[m\gamma_0 \operatorname{sh}(\gamma_0(h-c)) - \varkappa^2 \operatorname{ch}(\gamma_0(h-c))] - \Delta_0(u)\rho_0\gamma_0 \operatorname{sh}(\gamma_0(h-c)),$$

$$u^2 = (\alpha^2+\beta^2)R^2,\ \rho_0 = \rho_1/\rho_2,\ m = Rg/b^2,$$

$$\sigma = \sigma^{\mathrm{II}}/\rho_1 b^2,\ \Delta V_z = \Delta V_z^*/R\omega,$$

$$h = H/R,\ c = C/R,\ x = x/R,\ y = y/R,$$

$$\gamma_i = u^2 - \varepsilon_i^2\varkappa^2,\quad i = 0, 1, 2,\ \varkappa^2 = \omega^2R^2/b^2,$$

$$\varepsilon_0^2 = b^2/a^2,\ \varepsilon_1^2 = (1-2\nu)/2(1-\nu),\ \varepsilon_2^2 = 1,\ \gamma^2 = u^2 - \varkappa^2/2.$$

Here ρ_1, ρ_2 are the densities of the elastic medium and the liquid; b, a are the velocity of the transverse waves in the elastic layer and the velocity of sound in the liquid; C, H are the thicknesses of the elastic and the liquid media; g is the acceleration due to gravity; $\omega = 2\pi f$ is the frequency of the vibrations from the source; ν is Poisson's coefficient for an elastic medium; and R is the characteristic dimension of the vibrational source (for a circular source, it is the radius). Selection of the profiles Γ_1 and Γ_2 (Γ_1 is located above Γ_2) which coincide with the real axis and the real negative poles K(u) and M(u) from above and the real positive poles from below is considered in [5].

One can conclude from an analysis of the dispersion relations (1.2) that the functions K(u) and M(u) have equal poles (there is a finite number of real poles). At infinity $K(u) \sim O(|u|^{-1}),\ u \to \pm\infty$.

2. If the vibrational source has the form of a circle with an axially distributed load, then (1.1) transforms into

$$\int_\Gamma K(u) J_0(ur)\, u\, du \int_0^1 \Delta V_z(\rho) J_0(u\rho)\rho\, d\rho = f(r),\quad 0 \leqslant r \leqslant 1, \tag{2.1}$$

$$f(r) = \sigma(r) + \int_\Gamma M(u) J_0(ur)\, u\, du \int_0^1 \Delta\sigma(\rho) J_0(u\rho)\rho\, d\rho,\ r = \sqrt{x^2+y^2} \geqslant 0$$

($J_0(x)$ is a Bessel function). The contour Γ coincides with the semiaxis at almost all points $[0, \infty)$, which is the right half of profile Γ_1.

An analysis of the properties of the functions K(u) and M(u) allows one to apply the technique of left-side regularization to solve Eq. (2.1) [6]. The function K(u) can be approximated by a similar function K*(u) with the following left-side factorization

$$K^*(u) = K_+(u)K_-(u), \tag{2.2}$$

where $K_+(u)$ and $K_-(u)$ are regular functions in the upper and lower half-planes, respectively.

One can use the results of [6] to find the continuation of the right-hand side of the integral equation (2.1), the function f(r) (r > 1), in the form

$$f(r) = -\frac{1}{2}\int_{\Gamma_0} K_+(u)\frac{H_0^{(2)}(ur)}{H_0^{(2)}(u)} Y_+(u)\, du,\quad r > 1, \tag{2.3}$$

where $H_0^{(2)}(x)$ is a Hankel function, and contour Γ_0 (from $-\infty$ to $+\infty$) represents contour Γ over the entire real axis and is symmetric relative to the origin. The function $Y_+(u)$ is regular in the upper half-plane and satisfies the following integral equation

$$Y_+(\alpha) + \frac{1}{4\pi^2}\int_{\Gamma_{11}} \frac{du}{K_+(u)(u-\alpha)} \int_{\Gamma_{22}} \frac{K_+(t)\,\Theta(t,u)\,Y_+(t)}{t^2-u^2}\, dt = \frac{1}{2\pi i}\int_{\Gamma_{11}} \frac{H_0^{(2)}(u)F(u)}{K_+(u)(u-\alpha)} u\, du. \tag{2.4}$$

Here, α lies above contour Γ_{11} (from $-\infty$ to $+\infty$), which is, in turn, situated above Γ_{22} (from $-\infty$ to $+\infty$); contours Γ_{11} and Γ_{22} are obtained by deforming Γ_0 at the pole of regularity for the function in the integral; F(u) is the Fourier transformation of f(r)($r \leq 1$) on the right-hand side of Eq. (2.1);

$$\Theta(t,u) = -\pi i u H_0^{(2)}(u) J_0(u)\left[u\frac{J_1(u)}{J_0(u)} - t\frac{H_1^{(2)}(t)}{H_0^{(2)}(t)}\right] + t + u.$$

Omitting the integration contours Γ_{11} and Γ_{22} in the lower half-planes and intersecting a finite number of poles $K_+(\alpha)$ and zeros $K^{-1}_{+}(\alpha)$, the solution of (2.4) reduces to the solution of a system of linear algebraic equations in terms of the values of $Y_+(\alpha)$ at the poles $K_+(\alpha)$ which are intersected by contour Γ_{22}. Hence, we can use the estimation in [5] to neglect the integrals in terms of the deformed contours. One can show that (2.4) is nearly the Fredholm equation and that it reduces to an integral with a completely continuous integral operator. Hence, the obtained linear algebraic system can be effectively solved by numerical methods (in particular, Gauss' method).

A similar deformation of contour Γ_0 in the lower half-plane can be used to expand the right-hand side of Eq. (2.1) – function f(r) (r > 1) – into a series of the same values of $Y_+(\alpha)$ at the poles $K_+(\alpha)$ determined from the above algebraic system. Taking into account the right-hand side of Eq. (2.1) – function f(r) (r > 1) – one can obtain the unknown change in the velocity of the edges of the vibrational source through a dual transformation of the integral operator from (2.1).

3. We will assume that $\Delta\sigma(\rho) \equiv 0$, which corresponds to the case when forces that are equal in quantity and opposite in direction are applied to the upper and lower boundaries of the vibrational source. In this case

$$f(r) \equiv \sigma(r),\ r \leqslant 1. \tag{3.1}$$

Then, the extension of the right-hand side – function f(r) (r > 1) – is the stress $\sigma(r)$ at the boundary interface of the medium outside of the region occupied by the vibrational source.

A computer was used to determine the real singularities of the function K(u), and the following approximation function was obtained

$$K(u) \approx K^*(u) \equiv \frac{1}{\sqrt{u^2+B^2}} \prod_{i=1}^{N} \frac{u^2-\xi_i^2}{u^2-\eta_i^2}, \tag{3.2}$$

where among ξ_i, η_i there exist all real zeros $\xi_j (j = 1, 2, \ldots, N_1)$ and poles $\eta_j (j = 1, 2, \ldots, N_2)$ of the function K(u), while $N_1 < N$, $N_2 < N$. One obtains the required decrease at infinity by introducing $\sqrt{u^2 + B^2}$ into (3.2). The value of B is elected from the condition of the best approximation and from the condition of the infinitesimal character of the integrals in terms of the deformed contours (when (2.4) is reduced to an algebraic system). For our purposes, the optimal value is B = 10. An analysis of $\sigma(r)$ when $r \to 1$ allows one to conclude that the stress on the contour of the vibrational source ($z = -c$, $r \to 1 \pm 0$) changes in a finite manner ($\sigma(1 + 0) - \sigma(1 - 0) = \text{const}$).

We analyzed acoustic loads $\sigma(r)$ (r > 1) along the line of the boundary interface between the two media outside the region occupied by the vibrational source. In Fig. 1, one can see the dimensionless amplitude of the complex quantity $\sigma(r)$ as a function of the distance r from the center of vibrational source at r = 1, and the curves 1-6 correspond to $\kappa^2 = 1, 2, \ldots, 6$. Figure 1 also reflects the amplitude-frequency dependence of σ. For r = 1.1 it is evident that the absolute value of $\sigma(r)$ has a maximum when $\kappa^2 = 1$ and a minimum when $\kappa^3 = 3$. The values $f(r) = \sigma(r) = 1$ $(r \leqslant 1)$, $h = 5$, $c = 0{,}1$, $\rho_0 = 1{,}1$, $\varepsilon_0 = 0.66$, and $\nu = 0.3$ were used for making calculations.

4. In contrast to the previous problem which concerned an analysis of acoustic characteristics, we will now consider a similar problem where an ice field modeled by Kirchhoff's plate covers a layer of heavy, ideal liquid. This model, which only takes into account warping vibrations of the ice, well describes the process of excitation and propagation of surface gravitational waves [1].

Integral transformations can be used to reduce the problem to an equation equivalent to (1.1), where

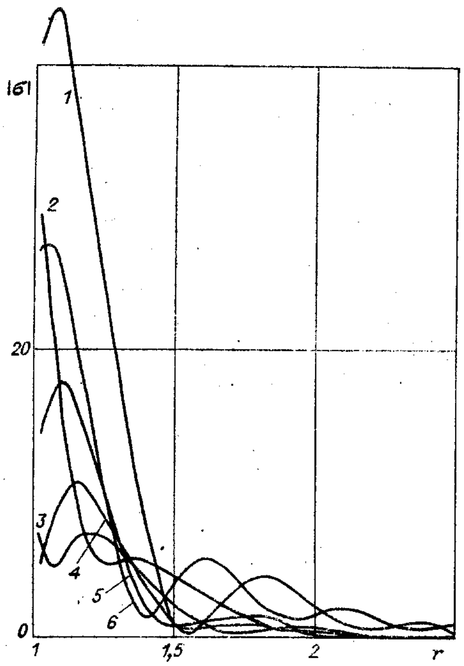

Fig. 1

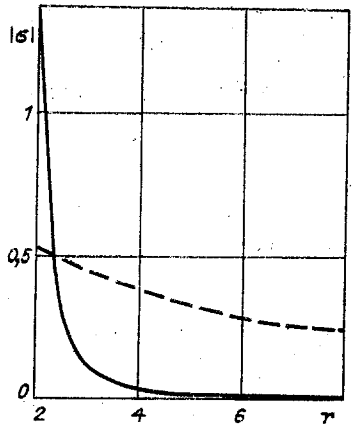

Fig. 2

$$K(u) = i\rho_0 D(u^4 - \lambda^2)[m\gamma_0 \operatorname{sh}(\gamma_0(h-c)) - \varkappa^2 \operatorname{ch}(\gamma_0(h-c))]/\Delta(u),$$
$$M(u) = [m\gamma_0 \operatorname{sh}(\gamma_0(h-c)) - \varkappa^2 \operatorname{ch}(\gamma_0(h-c))]/\Delta(u),$$
$$\Delta(u) = [D(u^4 - \lambda^2) + m]\gamma_0 \operatorname{sh}(\gamma_0(h-c)) - \varkappa^2 \operatorname{ch}(\gamma_0(h-c)),$$
$$D = \rho_0 c^3/6(1-\nu),\ \lambda^2 = 6(1-\nu)R^2\omega^2/c^2b^2, \tag{4.1}$$

and $K(u) \sim O(1)$ when $u \to \pm\infty$.

Similarly, in the previous solution of Eq. (1.1) [or after the transformation of (2.1) with the right-hand side of (3.1)], which was regularized by left-side factorization, the Fourier transformation can be used on the right-hand side, whose values in the range $r > 1$ are given by Eq. (2.3).

Numerical analysis was done of the solution for a heavy, incompressible liquid. The absolute values of the normal stresses at the boundary interface outside the region occupied by the vibrational source are given in Fig. 2, where the dashed line indicates the behavior of the shear forces due to the excitable gravitational mode when the ice is modeled as a thin plate, and the solid line shows the amplitude of the acoustic pressure on the ice (modeled by an elastic layer) as function of the radial coordinate r. Figure 2 can be used to estimate the contribution which the different components of the total normal pressure make to the dynamic action on the ice. Calculations were done for the dimensionless parameters indicated above and for $\kappa^2 = 0.1$.

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